J. Fluid Mech. (1987), vol. 176, pp. 311–332
Printed in Great Britain

The Benjamin–Feir instability of a deep-water Stokes wavepacket in the presence of a non-uniform medium

By MARIUS GERBER

Department of Ocean Engineering, Stellenbosch University, Stellenbosch 7600, South Africa

The influence of a non-uniform medium on the Benjamin–Feir instability of weakly nonlinear deep-water waves has been investigated, and an approach via a suitable nonlinear Schrödinger equation was adopted. For the derivation of the relevant cubic Schrödinger equation, the approach of Yuen & Lake (1975) was followed and an applicable dispersion relation and energy equation was derived by the averaged Lagrangian technique. With the assumption that the lengthscale of current variation is much greater than the lengthscale of the wavepacket, a cubic Schrödinger equation with slowly varying coefficients is obtained. Three different examples of non-uniform media are treated: (i) waves on a current with variation along the stream; (ii) waves on a shear current; and (iii) long deep-water gravity waves interacting with shorter waves.

1. Introduction

Since the discovery by Benjamin & Feir (1967) that weakly nonlinear surface gravity waves are unstable to sideband perturbations, rapid advances in the understanding of the behaviour of such deep-water waves have taken place. In particular, investigation of the properties of the nonlinear Schrödinger equation, which is the evolution equation for the slowly varying envelope of the carrier wave, has led to applications in many different areas. The effects of variable depth have been studied by Djordjevic & Redekopp (1978), who deduced a cubic Schrödinger equation with variable coefficients for mild bottom slopes. Turpin, Benmoussa & Mei (1983) extended the analysis of Djordjevic & Redekopp (1978) to include the effects of a slowly varying current. A cubic Schrödinger equation was also found for the wave envelope, the coefficients now being a function of the topography as well as the current. Smith (1976) also deduced a cubic Schrödinger equation in an attempt to explain giant waves as encountered in the Agulhas current on the east coast of South Africa.

The Benjamin–Feir instability criterion can also be recovered from the cubic Schrödinger equation, as shown by Yuen & Lake (1978), Stuart & DiPrima (1978), West (1981) and other workers. Even better agreement between theory and experiment was obtained by Dysthe (1979) who calculated an improvement to the Benjamin–Feir instability criterion using a fourth-order envelope equation. The dominant new effect introduced by this equation is the wave-induced mean flow which gave rise to a Doppler shift in the frequency of the carrier wave, and a corresponding down-shift in experimental values, in agreement with that found by Lake *et al.* (1977). Janssen (1983), however, argued that since the frequency shift associated with this fourth-order equation is periodic in space and time, it must be due to other effects, such as dissipation.

All the parameters describing a weakly nonlinear wavetrain can be influenced by large-scale current variations. For a slowly varying current $U(\boldsymbol{x}, t)$, changes in the flow velocity will cause corresponding variations in the apparent frequency and, as mentioned, a Doppler shift in the frequency of the carrier wave. Energy exchanges between the current and the wavetrain will also lead to amplification or dissipation of the wave amplitude. From the results of Dysthe (1979) and Janssen (1983), a reasonable assumption would thus be that a shift in the sideband growth rates can be expected for weakly nonlinear waves in an inhomogeneous medium. This is contrary to the results of Smith (1976) who found that modulations are absent at a caustic when surface waves are reflected by a shear current. Peregrine (1976) also mentioned that a current will not influence the instability criterion, but no analysis in proof is presented. In a numerical study by Turpin *et al.* (1983), however, these authors show that a current may well influence the stability of nonlinear waves. A positive supercritical current was found to give rise to stable Stokes waves in deep water, while a negative subcritical current will cause all Stokes waves to become unstable.

In this paper we intend to investigate the influence of a steady current on the Benjamin–Feir instability criterion of surface waves. A cubic Schrödinger equation in the presence of a current will form the basis of our analysis. For the analysis the approach of Yuen & Lake (1975) is followed and the governing equation is derived from a suitable variational principle, leading, as an intermediate step, to a dispersion relation and an energy equation (§2). In §3 the desired cubic Schrödinger equation will be derived from the above two relations. The derivation involves three small parameters: the usual wave slope ϵ; the spectral bandwidth Δ; and a third parameter introduced by the current. This means that asymptotic limits may be identified, and some aspects of the asymptotic solutions so introduced are discussed in §4. Sections 4.1 and 4.2 are devoted to the derivation, and discussion, of the instability criterion that would apply in the presence of a current. A steady current, varying with distance along the stream, is treated in §4.1, while a shearing current is considered in §4.2. Since the results of §4 are also expected to apply for long waves interacting with short waves, some aspects of such interactions are discussed in §4.3. Finally, §5 summarizes our conclusions.

2. The variational principle and modulation equations

The general use of a variational principle, leading to the calculation of a dispersion relation and an energy equation for finite-amplitude waves, is well documented in the literature. The interested reader is referred to Whitham (1974) for an especially lucid presentation. For a non-uniform medium, varying slowly in space and time, the 'average variational principle' (Whitham 1960, 1962, 1965) will also apply, with the difference that the averaged Lagrangian may depend explicitly on $\boldsymbol{x}$ and t, as well as through the amplitude and phase functions.

For waves on still water, Yuen & Lake (1975) calculated an averaged Lagrangian, $\langle L\rangle$, to second order in both wave slope ϵ and bandwidth Δ, using the Lagrangian given by Luke (1967):

$$L = \int_0^{\zeta} [\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + gz]\,\mathrm{d}z,$$

where ϕ is the velocity potential, and the free surface of the fluid is given by ζ.

Variation of $\langle L\rangle$ with respect to the amplitude and phase then generate, respectively, the dispersion relation

$$\sigma = (gk)^{\frac{1}{2}}\left[1+\tfrac{1}{2}k^2a^2+\frac{1}{8}\frac{a_{xx}}{k^2a}\right]$$

(note the correction in the last term when compared with Yuen & Lake 1975) and energy equation

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}(C_{g_0}E) = 0; \quad C_{g_0} = \frac{\mathrm{d}\sigma_1}{\mathrm{d}k}, \quad \sigma_1 = (gk)^{\frac{1}{2}}.$$

For waves in the presence of a current, a Lagrangian corresponding to the integral of the Bernoulli equation, superimposed on a current $U(x,t)$, then seems a logical extension. With the assumption that the unsteady, non-uniform current motion on a level mean surface can be represented as irrotational motion so that a velocity potential $\boldsymbol{U} = \nabla\Phi$ can be defined, the Lagrangian becomes

$$L = \int_0^{\zeta(x,y,t)} [(\phi+\Phi)_t+\tfrac{1}{2}\{\nabla(\phi+\Phi)\}^2+gz]\,\mathrm{d}z. \tag{2.1}$$

The effect of vorticity will be reflected in the dispersion relation to be generated by the averaged Lagrangian. Weak vorticity can be expected to have a correspondingly weak influence on the eventual cubic Schrödinger equation, so that, to second order, the assumption of irrotationality can be justified.

The averaged Lagrangian for weakly nonlinear waves on a steady, non-uniform, one-dimensional current $\boldsymbol{U} = [U(x),0,0]$ can be calculated from (2.1). Since the analysis will be confined to waves in deep water, the assumption of Lighthill (1965), that the mean height and mean velocity will play a negligible role, will apply. It should be noted, however, that this assumption does not hold for long waves and a pseudo-phase must be defined as was done by Whitham (1974). The expression obtained for the averaged Lagrangian is

$$\langle L\rangle = -\frac{1}{4}\frac{(\omega-kU)^2}{k}a_1^2+\tfrac{1}{4}g(a_1^2+\tfrac{1}{2}k^2a_1^4). \tag{2.2}$$

This is essentially the same as the expression obtained by Yuen & Lake (1975) when their analysis is only carried through to $O(\epsilon^2,\Delta^0)$; that is when higher-order dispersion is not taken into account, and when $(\omega-kU)$ in (2.2) is replaced by the intrinsic frequency σ, which is the correct parameter in the absence of a current. Higher-order dispersive effects can now be introduced in the analysis by using the idea of pseudo-differential operators as suggested by Whitham (1974, p. 526). The eventual expression obtained for the averaged Lagrangian (in physical variables x and t) is

$$\bar{L} = [\omega-(gk)^{\frac{1}{2}}-kU]\,a^2-\tfrac{1}{4}(gk)^{\frac{1}{2}}k^2a^4+\frac{(gk)^{\frac{1}{2}}a_x^2}{8k^2}, \tag{2.3}$$

where terms of $O(\epsilon^2,\Delta^2)$ have been retained. It should be noted that this expression is not a true averaged Lagrangian in that variation with respect to the wavenumber and frequency will not render the correct expression for the wave action and wave-action flux. However, (2.3) will produce the correct dispersion relation to the desired order. For this reason the notation $\langle L\rangle$, which is reserved for the true averaged Lagrangian, was not used in (2.3).

The modulation equations are finally obtained from the variational principle

$$\delta \iint \langle L\rangle(\omega, k, a, a_x)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Variation of (2.3) with respect to the amplitude a then gives the dispersion relation

$$\omega = (gk)^{\frac{1}{2}}\left[1+\tfrac{1}{2}k^2a^2+\frac{1}{8}\frac{a_{xx}}{k^2a}\right]+kU. \tag{2.4}$$

The term $\frac{1}{2}(gk)^{\frac{1}{2}}k^2a^2$ is the well-known Stokes correction for nonlinearity, while the term $\frac{1}{8}(gk)^{\frac{1}{2}}a_{xx}/k^2a$ accounts for the first correction due to higher-order dispersion. The last term is just the Doppler shift due to the influence of the current. Finally, variation of (2.2) with respect to the phase θ gives the desired energy equation

$$\frac{\partial}{\partial t}\left[\frac{a^2}{\sigma_1}\right]+\frac{\partial}{\partial x}\left[\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}+U\right)\frac{a^2}{\sigma_1}\right] = 0. \tag{2.5}$$

Here $\sigma_1 = (gk)^{\frac{1}{2}} = \omega - kU$ is the linear dispersion relation in the presence of a current. Equation (2.5) is the well-known wave-action equation which, for $a \equiv a(x,t)$, $k \equiv k(x,t)$ and $\sigma \equiv \sigma(k)$, can be expanded to give

$$\frac{\partial}{\partial t}a^2+\frac{\partial}{\partial x}\left[\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}+U\right)a^2\right]+\left[\frac{k}{\sigma_1}\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}\right]\frac{\partial U}{\partial x}a^2 = 0. \tag{2.6}$$

For waves in deep water, (2.6) can be further simplified to

$$\frac{\partial a}{\partial t}+\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}\frac{\partial a}{\partial x}+\tfrac{1}{2}a\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}\right)+U\frac{\partial a}{\partial x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}a = 0. \tag{2.7}$$

Equations (2.4) and (2.7) are the two equations required in the next section to derive a cubic Schrödinger equation which describes the evolution of a nonlinear wavepacket in the presence of a current.

3. The cubic Schrödinger equation for a non-uniform medium

The dispersion relation and energy equation for Stokes waves on a non-uniform flow were given by (2.4) and (2.7). In this section we shall show that both these expressions will contribute to the derivation of a dissipative type of cubic Schrödinger equation which will apply in the presence of a current. This equation, which will be called the Modified Cubic Schrödinger Equation (MCSE), is the evolution equation for the amplitude of a nonlinear wavetrain superimposed on a current. Turpin *et al.* (1983) derived a similar equation (when the deep-water limit of their equation (2.22) is taken) using the method of multiple scales. Alternatively, a simpler, more heuristic, derivation would be possible by identifying the appropriate interaction terms of linear theory, and suitably modifying the still-water cubic Schrödinger equation. A two-dimensional extension to the MCSE will also be presented in this section. Since two ordering parameters were introduced in the calculation of the dispersion relation, both parameters will be reflected in different terms in the MCSE. In addition, the derivation will introduce a third ordering parameter π, the ratio of the wave group length to the lengthscale of the current. For very small values of π, the still-water cubic Schrödinger equation will be recovered.

3.1. *Steady current, varying with distance along the stream*

For the derivation, the narrow-spectrum approach, like that of Phillips (1981), will be followed. That is, for a slowly varying wavetrain, the only significant contributions are assumed to come from a narrow spread of wavenumbers and frequencies around a central wavenumber k_0 and frequency ω_0. In spectral terms, if the surface displacement of the wave group on still water is given by

$$\zeta(x,t) = \int a(k)\,\mathrm{e}^{\mathrm{i}\theta(x,t)}\,\mathrm{d}k,$$

where, in one dimension, $\theta = kx - \omega t$, and $\omega = \omega(k)$, this implies that $a(k)$ will be small except when $k - k_0 \sim O(\Delta)$, $\Delta \ll 1$. Here the spectral bandwidth, $\Delta = \delta k/k$, the ratio of individual wavelength to the group length, is the parameter used to describe the narrowness of the spectrum. The phase function $\theta(x,t)$ will also reflect the slow variation

$$\theta(x,t) = \theta_0 + \Theta(X,T); \quad k = k_0 + \Delta\kappa, \quad \omega = \omega_0 + \Delta\Omega,$$

where the slow variables are $X = \Delta x$, $T = \Delta t$ and $\Delta \ll 1$. In the presence of a steady current $\boldsymbol{U} = [U(x), 0, 0]$ the dispersion relation becomes $\omega = \sigma(k) + kU$, with the fast variation in phase $\theta_0(x,t) = k_0 x - \omega_0 t$, while the slow phase variation is

$$\begin{aligned}\Theta(X,T) &= \Delta\kappa x - \Delta\Omega t \\ &= \Delta\kappa x - (\omega - \omega_0)\,t \\ &= \Delta\kappa x - [(\sigma - \sigma_0) + \Delta\kappa U]\,t \\ &= \kappa X - [\Sigma + \kappa U]\,T; \quad \Delta\Sigma = \sigma - \sigma_0, \quad X = \Delta x, \quad T = \Delta t,\end{aligned}$$

so that

$$\Delta\frac{\partial\Theta}{\partial X} = \Delta\kappa, \tag{3.1}$$

$$\Delta\frac{\partial\Theta}{\partial T} = -[\sigma(k) - \sigma_0 + \Delta\kappa U]. \tag{3.2}$$

The surface displacement of the group in the presence of the slowly varying current $U(X)$ is then

$$\zeta(x,t) = \int a(k, U(X))\,\mathrm{e}^{\mathrm{i}\theta(x,t)}\,\mathrm{d}k,$$

which, for the narrow-spectrum approach can be written as

$$\zeta = \exp\{\mathrm{i}(k_0 x - [\sigma_0 + k_0 U]\,t\} \int a(k, U(X)) \exp\{\mathrm{i}[\kappa X - (\Sigma + \kappa U)\,T]\}\,\mathrm{d}k.$$

The envelope of the group is given by

$$A(X,T) = \int a(k, U) \exp\{\mathrm{i}[\kappa X - (\Sigma + \kappa U)\,T]\}\,\mathrm{d}k,$$

so that the rate of change of the envelope is

$$\begin{aligned}\frac{\partial A}{\partial T} &= \int \left\{\frac{\partial a}{\partial T} - \mathrm{i}a[\Sigma + \kappa U]\right\} \exp\{\mathrm{i}[\kappa X - (\Sigma + \kappa U)\,T]\}\,\mathrm{d}k \\ &\equiv \int \left\{\frac{\partial a}{\partial T} + \mathrm{i}a\frac{\partial\Theta}{\partial T}\right\} \mathrm{e}^{\mathrm{i}\Theta}\,\mathrm{d}k,\end{aligned}$$

reflecting a slow change in the amplitude and phase functions.

For nonlinear waves on a current, and with second-order frequency dispersion, the intrinsic part of the dispersion relation (2.4) gives

$$\sigma(k,a) = (gk)^{\frac{1}{2}}\left[1+\tfrac{1}{2}k^2a^2+\frac{1}{8}\frac{a_{xx}}{k^2a}\right]. \tag{3.3}$$

This can be written as

$$\sigma(k,a) \equiv \sigma_1(k)+\sigma_2(k)\,a^2-\frac{1}{2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}\frac{a_{xx}}{a},$$

with $\sigma_1 = (gk)^{\frac{1}{2}}$ and $\sigma_2 = \frac{1}{2}\sigma_1 k^2$. Since the spectrum was assumed narrow, $\sigma(k)$ can be expanded in a Taylor series about the central wavenumber k_0. The linear dispersion relation $\sigma_1(k)$ then becomes

$$\sigma_1(k) = \sigma_1(k_0)+\Delta\kappa\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\tfrac{1}{2}(\Delta\kappa)^2\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+O(\Delta^3). \tag{3.4}$$

On substitution of (3.3) and (3.4) in (3.2), the slow rate of change in phase of nonlinear waves in the presence of a current is

$$\Delta\frac{\partial\Theta}{\partial T}+\Delta\kappa\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\tfrac{1}{2}(\Delta\kappa)^2\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\sigma_2(k_0)\,a^2-\frac{1}{2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\frac{a_{xx}}{a}+\Delta\kappa U = 0. \tag{3.5}$$

For $X = \Delta x, T = \Delta t$, and using (3.1), this gives

$$\Delta\frac{\partial\Theta}{\partial T}+\Delta\frac{\partial\Theta}{\partial X}\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\tfrac{1}{2}\Delta^2\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\sigma_2(k_0)\,a^2-\tfrac{1}{2}\Delta^2\frac{1}{a}\frac{\partial^2 a}{\partial X^2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\Delta\frac{\partial\Theta}{\partial X}U = 0. \tag{3.6}$$

The amplitude (energy) equation was derived from the wave-action equation in the previous section. The form (2.7) will be used in this derivation to reflect the slow change in amplitude of the wavepacket. The terms involving derivatives of the linear dispersion relation in (2.7) can be substituted for from (3.4) in the following manner:

$$\left.\begin{aligned}\frac{\mathrm{d}\sigma_1}{\mathrm{d}k} &= \frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\Delta\kappa\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\ldots,\\ \frac{\mathrm{d}}{\mathrm{d}\kappa}\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)\right]+\Delta\kappa\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\right]+\Delta\frac{\partial\kappa}{\partial x}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\ldots.\end{aligned}\right\} \tag{3.7}$$

The first two terms on the right-hand side of (3.7) are due to the influence of the current, so that three different small parameters will contribute to the eventual expression for the middle term of (2.7). This is because the current variation introduces another lengthscale, and the small parameter $\pi = L_g/L_c$, the ratio of the group length to the current lengthscale, may be introduced. The middle term of (2.7) then gives to the desired order

$$a\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}\right) \sim a\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)\right]+a\Delta\kappa\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\right]+a\Delta\frac{\partial\kappa}{\partial x}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0). \tag{3.8}$$

The three terms on the right-hand side of (3.8) are respectively of order $\epsilon\Delta\pi$, $\epsilon\Delta^2\pi$ and $\epsilon\Delta^2$, so that for $\pi \ll 1$ the second term will be smaller than the other two and can be neglected. Equation (2.7) now becomes

$$\frac{\partial a}{\partial t}+\left\{\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\Delta\kappa\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\right\}\frac{\partial a}{\partial x}+\tfrac{1}{2}a\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)\right]+\Delta\frac{\partial\kappa}{\partial x}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\right\}+U\frac{\partial a}{\partial x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}a = 0, \tag{3.9}$$

or when the slow variables are identified:

$$\Delta\frac{\partial a}{\partial T}+\Delta\frac{\partial a}{\partial X}\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+\Delta U\frac{\partial a}{\partial X}+\tfrac{1}{2}\Delta\frac{\mathrm{d}}{\mathrm{d}X}\left[\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)\right]a+\tfrac{3}{4}\Delta\frac{\mathrm{d}U}{\mathrm{d}X}a$$
$$+\Delta^2\frac{\partial\Theta}{\partial X}\frac{\partial a}{\partial X}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)+\tfrac{1}{2}\Delta^2 a\frac{\partial^2\Theta}{\partial X^2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)=0. \quad (3.10)$$

The terms retained are $O(\epsilon\Delta\pi, \epsilon\Delta^2)$.

A complex wave amplitude, slowly modulating the carrier wave, can now be defined as

$$A(X,T)=a(X,T)\exp\left[\mathrm{i}\Theta(X,T)\right],$$

where the governing equations for phase and amplitude are given by (3.6) and (3.10) respectively. We note that

$$\mathrm{i}\frac{\partial A}{\partial t}=\Delta\left[\mathrm{i}\frac{\partial a}{\partial T}-\frac{\partial\Theta}{\partial T}a\right]\mathrm{e}^{\mathrm{i}\Theta},$$
$$\mathrm{i}\frac{\partial A}{\partial x}=\Delta\left[\mathrm{i}\frac{\partial a}{\partial X}-\frac{\partial\Theta}{\partial X}a\right]\mathrm{e}^{\mathrm{i}\Theta},$$
$$\frac{1}{2}\frac{\partial^2 A}{\partial x^2}=\Delta^2\left[\mathrm{i}\frac{\partial\Theta}{\partial X}\frac{\partial a}{\partial X}+\frac{1}{2}\frac{\partial^2\Theta}{\partial X^2}a+\frac{1}{2}\frac{\partial^2 a}{\partial X^2}-\frac{1}{2}\left(\frac{\partial\Theta}{\partial X}\right)^2a\right]\mathrm{e}^{\mathrm{i}\Theta},$$
$$|A|^2A=a^3\,\mathrm{e}^{\mathrm{i}\Theta},$$

so that if $a\,\mathrm{e}^{\mathrm{i}\Theta(X,T)}$ multiplied with (3.6), is subtracted from $\mathrm{i}\,\mathrm{e}^{\mathrm{i}\Theta(X,T)}$ multiplied with (3.10), the resulting combination is

$$\mathrm{i}\left[\frac{\partial A}{\partial t}+\left\{\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)+U\right\}\frac{\partial A}{\partial x}+\left\{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}\sigma_1}{\mathrm{d}k}(k_0)\right]+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}\right\}A\right]$$
$$+\frac{1}{2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}k^2}(k_0)\frac{\partial^2 A}{\partial x^2}-\sigma_2(k_0)|A|^2A=0. \quad (3.11)$$

On substitution of the values for σ_1 and σ_2 the MCSE can finally be written as

$$\frac{\partial A}{\partial t}+(C_{g_0}+U)\frac{\partial A}{\partial x}+\left(\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}\right)A+\tfrac{1}{8}\mathrm{i}\frac{\sigma_0}{k_0^2}\frac{\partial^2 A}{\partial x^2}+\tfrac{1}{2}\mathrm{i}\sigma_0k_0^2|A|^2A=0, \quad (3.12)$$

since the group velocity $C_{g_0}=\tfrac{1}{2}\sigma_0/k_0$.

3.2. *Two-dimensonal currents*; $\boldsymbol{U}=(U,V,0)$

The procedure of §3.1 can also be followed to obtain the form of the MCSE when in the presence of a two-dimensional current $\boldsymbol{U}=(U(x),V(y),0)$. In this case the governing equation for the wave amplitude will be the two-dimensional deep-water counterpart of (2.6), or the so-called radiation stress equation (Longuet-Higgins & Stewart 1964)

$$\frac{\partial E}{\partial t}+\boldsymbol{\nabla}\cdot\left[\left(\frac{\partial\sigma}{\partial\boldsymbol{k}}+\boldsymbol{U}\right)E\right]+\boldsymbol{S}:\boldsymbol{\gamma}=0. \quad (3.13)$$

The rate-of-strain tensor for the mean flow in (3.13) is

$$\boldsymbol{\gamma}=\begin{bmatrix}\dfrac{\partial U}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial U}{\partial y}+\dfrac{\partial V}{\partial x}\right)\\ \dfrac{1}{2}\left(\dfrac{\partial U}{\partial y}+\dfrac{\partial V}{\partial x}\right) & \dfrac{\partial V}{\partial y},\end{bmatrix}$$

while, for a coordinate system orientated at an angle ϕ to the propagation direction, the radiation stress tensor is given by

$$\boldsymbol{S} = \tfrac{1}{2}E\begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}.$$

Since $E = \frac{1}{2}\rho g a^2$, (3.13) can be written as

$$\frac{\partial a}{\partial t}+\left(\frac{\partial \sigma}{\partial l}+U\right)\frac{\partial a}{\partial x}+\left(\frac{\partial \sigma}{\partial m}+V\right)\frac{\partial a}{\partial y}+\tfrac{1}{2}a\frac{\partial}{\partial x}\left(\frac{\partial \sigma}{\partial l}+U\right)+\tfrac{1}{2}a\frac{\partial}{\partial y}\left(\frac{\partial \sigma}{\partial m}+V\right)$$
$$+\tfrac{1}{4}\cos^2\phi\frac{\partial U}{\partial x}a+\tfrac{1}{4}\sin^2\phi\frac{\partial V}{\partial y}a+\tfrac{1}{4}\sin\phi\cos\phi\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial x}\right)a = 0, \quad (3.14)$$

where the dispersion relation $\sigma = \sigma(l, m)$.

To generate the higher-order dispersive terms, the linear dispersion relation $\sigma_1(l, m)$ must be expanded in a Taylor series about l_0, m_0. Thus

$$\sigma_1(l,m) = \sigma_1(l_0,m_0)+(l-l_0)\frac{\partial \sigma_1}{\partial l}(l_0,m_0)+(m-m_0)\frac{\partial \sigma_1}{\partial m}(l_0,m_0)$$
$$+\frac{1}{2}\left\{(l-l_0)^2\frac{\partial^2 \sigma_1}{\partial l^2}(l_0,m_0)+(m-m_0)^2\frac{\partial^2 \sigma_1}{\partial m^2}(l_0,m_0)+2(l-l_0)(m-m_0)\frac{\partial^2 \sigma_1}{\partial l\,\partial m}(l_0,m_0)\right\}.$$

The only other relation necessary for the derivation is the two-dimensional counterpart of (2.4). This is given by

$$\omega = (gk)^{\frac{1}{2}}\left[1+\tfrac{1}{2}k^2a^2+\frac{1}{8k^2a}\left(\frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial x\,\partial y}+\frac{\partial^2 a}{\partial y^2}\right)\right]+kU,$$

with $k = |\boldsymbol{k}| = (l^2+m^2)^{\frac{1}{2}}$. Following the procedure of the previous section, the form of the MCSE, equivalent to (3.11), is found to be

$$\mathrm{i}\left[\frac{\partial A}{\partial t}+\left(\frac{\partial \sigma_1}{\partial l}\Big|_0+U\right)\frac{\partial A}{\partial x}+\left(\frac{\partial \sigma_1}{\partial m}\Big|_0+V\right)\frac{\partial A}{\partial y}+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \sigma_1}{\partial l}+U\right)A+\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \sigma_1}{\partial m}+V\right)A\right.$$
$$\left.+\tfrac{1}{4}\cos^2\phi\frac{\partial U}{\partial x}A+\tfrac{1}{4}\sin^2\phi\frac{\partial V}{\partial y}A+\tfrac{1}{4}\sin\phi\cos\phi\left(\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\right)A\right]$$
$$+\frac{1}{2}\frac{\partial^2\sigma_1}{\partial l^2}\Big|_0\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2\sigma_1}{\partial l\,\partial m}\Big|_0\frac{\partial^2 A}{\partial x\,\partial y}+\frac{1}{2}\frac{\partial^2\sigma_1}{\partial m^2}\Big|_0\frac{\partial^2 A}{\partial y^2}-\sigma_2\Big|_0|A|^2A = 0. \quad (3.15)$$

This is the most general form. For a constant dominant wave vector $\boldsymbol{k} = (k_0, 0)$, and on substitution for σ_1 and σ_2, (3.15) becomes

$$\mathrm{i}\left[\frac{\partial A}{\partial t}+(C_{g_0}+U)\frac{\partial A}{\partial x}+\left(\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}\right)A\right]-\frac{\sigma_0}{8k_0^2}\frac{\partial^2 A}{\partial x^2}+\frac{\sigma_0}{4k_0^2}\frac{\partial^2 A}{\partial y^2}-\tfrac{1}{2}\sigma_0k_0^2|A|^2A = 0.$$

Note that the one-dimensional form of the MCSE can be recovered from (3.15) with the restrictions $\boldsymbol{U} = [U(x), 0, 0]$ and $\boldsymbol{C}_{g_0} = (\mathrm{d}\sigma_1/\mathrm{d}l, 0, 0)$.

3.3. *Steady current, varying across the stream*

The form of the MCSE for waves on a shearing current, $\boldsymbol{U} = [0, V(x), 0]$, and with an angle ϕ between the wave crests and the x-axis, can be obtained as a simplification of (3.15). The form of the equation obtained is

$$\mathrm{i}\left[\frac{\partial A}{\partial t}=\frac{\mathrm{d}\sigma_1}{\mathrm{d}l}\Big|_0\frac{\partial A}{\partial x}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}l}\right)+\tfrac{1}{4}\sin\phi\cos\phi\frac{\mathrm{d}V}{\mathrm{d}x}A\right]+\frac{1}{2}\frac{\mathrm{d}^2\sigma_1}{\mathrm{d}l^2}\Big|_0\frac{\partial^2 A}{\partial x^2}-\sigma_2\Big|_0|A|^2A = 0.$$

For $\sigma_1 = (gk)^{\frac{1}{2}}, \sigma_2 = \frac{1}{2}\sigma_1 k^2$ this is

$$\frac{\partial A}{\partial t}+C_{g_0}\cos\phi\frac{\partial A}{\partial x}+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(C_{g_0}\cos\phi)A+\tfrac{1}{4}\sin\phi\cos\phi\frac{\mathrm{d}V}{\mathrm{d}x}A+\tfrac{1}{8}\mathrm{i}\frac{\sigma_0}{k_0^2}\frac{\partial^2 A}{\partial x^2}+\tfrac{1}{2}\mathrm{i}\sigma_0 k_0^2|A|^2 A = 0, \quad (3.16)$$

since the component of the group velocity in the x-direction is $C_{g_0}\cos\phi$.

4. The Benjamin–Feir instability for a non-uniform medium

4.1. *Steady current, varying with distance along the stream*

The form of the MSCE that applies in this situation was given by (3.12). In the derivation three different ordering parameters were identified – the usual two still-water parameters: the wave slope $\epsilon = ka$, and the spectral bandwidth $\Delta = \lambda/L_g$, the ratio of wavelength to group length. In addition a third parameter $\pi = L_g/L_c$, the ratio of group length to the lengthscale of current variation, was introduced. The MCSE, (3.12), can be written as

$$\frac{\partial A}{\partial t}+[C_{g_0}(x)+U(x)]\frac{\partial A}{\partial x}+\left[\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}\right]A+\mathrm{i}\gamma(x)\frac{\partial^2 A}{\partial x^2}+\mathrm{i}\beta(x)|A|^2 A = 0, \quad (4.1)$$

where $\gamma = \frac{1}{8}\sigma_0/k_0^2$ and $\beta = \frac{1}{2}\sigma_0 k_0^2$. The nonlinear self-interaction term, the last term on the left-hand side of (4.1), is of order ϵ^3, while the dispersive term, the term involving $\gamma(x)$, is of order $\epsilon\Delta^2$. The first three terms are individually of order $\epsilon\Delta$. This leaves the two dissipative terms, with ordinary derivatives, which are easily shown to be of order $\epsilon\Delta\pi$. In the limit $\pi \ll \epsilon$ and $\pi \ll \Delta$, but not restricting the magnitude of ϵ and Δ with respect to each other, the cubic Schrödinger equation in the presence of a constant current $U = \tilde{U}$ is recovered. If the superscript ˜ is used to denote constant parameters (as would be found when $U = \tilde{U}$, or on still water when $U \equiv 0$) the coefficients of (4.1) will, in this case, be

$$\tilde{C}_{g_0} = \frac{1}{2}\frac{\tilde{\sigma}_0}{\tilde{k}_0}, \quad \tilde{\gamma} = \frac{1}{8}\frac{\tilde{\sigma}_0}{\tilde{k}_0^2}, \quad \tilde{\beta} = \tfrac{1}{2}\tilde{\sigma}_0\tilde{k}_0^2.$$

With the further restriction $\Delta \ll \epsilon \ll 1$, the nonlinear Stokes wave equation is obtained, the solution of which is usually given by

$$\begin{aligned} A(t) &= \tilde{a}_0 \exp[-\tfrac{1}{2}\mathrm{i}\tilde{\sigma}_0\tilde{k}_0^2\tilde{a}_0^2 t] \\ &\equiv \tilde{a}_0\exp[-\mathrm{i}\tilde{\beta}\tilde{a}_0^2 t]. \end{aligned} \quad (4.2)$$

In the limit $\epsilon \ll \Delta \ll 1$, the linear propagation of small-amplitude groups are described, while for $\epsilon \sim \Delta \ll 1$, the usual sech-profile soliton solution can be obtained from the inverse scattering transform.

To calculate the Benjamin–Feir instability for the particular form of inhomogeneous medium, the basic state to be perturbed will be a Stokes wave in the presence of a current. The governing equation is (4.1) in the limit when $\Delta \ll \pi \ll 1$ and $\Delta \ll \epsilon \ll 1$, but not restricting the wave slope ϵ relative to π. The resulting equation is

$$\frac{\partial A_1}{\partial t}+(C_{g_0}+U)\frac{\partial A_1}{\partial x}+\left(\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}\right)A_1+\mathrm{i}\beta(x)|A_1|^2 A_1 = 0. \quad (4.3)$$

The solution of (4.3) (to $O(\epsilon\Delta\pi, \epsilon^3)$) is

$$A_1(x,t) = \Phi_1\left(\int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t\right) \exp\left[-\frac{1}{2}\int_{x_0}^{x} P\,\mathrm{d}x\right]$$
$$\times \exp\left\{\mathrm{i}\left[\Phi_2\left(\int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t\right) - \left[\Phi_1\left(\int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t\right)\right]^2\right.\right.$$
$$\left.\left.\times \int_{x_0}^{x} \left(\frac{\beta \exp\left[\int_{x_0}^{x} P\,\mathrm{d}x\right]}{C_{g_0}+U}\right)\mathrm{d}x\right]\right\}, \tag{4.4}$$

with Φ_1 and Φ_2 arbitrary functions of their arguments, and where

$$P(x) = \frac{\dfrac{\mathrm{d}C_{g_0}}{\mathrm{d}x} + \dfrac{3}{2}\dfrac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}.$$

Equation (4.4) can be shown to simplify to (4.2) when $U = 0$.

The infinitesimal perturbation is taken as

$$B(x,t) = a_1(x,t) \exp\left[\mathrm{i}\Sigma\left(\int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t\right)\right]$$
$$\equiv a_1(x,t)\,\mathrm{e}^{\mathrm{i}\Sigma\eta},$$

where

$$\eta = \int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t$$

is the moving coordinate and Σ is the perturbation frequency.

On substitution of

$$A = [1 + B(x,t)]\,A_1(x,t)$$

into (4.1), the lowest-order linear equation for the perturbation is found as

$$\mathrm{e}^{\mathrm{i}\Sigma\eta}\left[\frac{\partial a_1}{\partial t} + (C_{g_0}+U)\frac{\partial a_1}{\partial x} - \mathrm{i}\gamma(x)\,\kappa^2 a_1\right] + \mathrm{i}\beta(x)\,|A_1|^2\,[\mathrm{e}^{\mathrm{i}\Sigma\eta} + \mathrm{e}^{-\mathrm{i}\Sigma\eta}]\,a_1 = 0, \tag{4.5}$$

where the perturbation wavenumber $\kappa = \Sigma/(C_{g_0}+U)$. If now (4.5) is multiplied by $a_1\,\mathrm{e}^{-\mathrm{i}\Sigma\eta}$, and the complex-conjugate of (4.5) is added to the resulting expression, the differential equation for the perturbation is obtained as

$$\frac{\partial}{\partial t}(a_1^2) + (C_{g_0}+U)\frac{\partial}{\partial x}(a_1^2) + 2\beta(x)\,|A_1|^2 \sin(2\Sigma\eta)\,a_1^2 = 0. \tag{4.6}$$

Clearly, exponential growth of a_1 will result if

$$\int \frac{\beta(x)\,|A_1|^2}{C_{g_0}+U} \sin 2\Sigma\eta\,\mathrm{d}\kappa < 0. \tag{4.7}$$

From (4.4) we find that

$$|A_1|^2 = |\Phi_1|^2 \exp\left[-\int_{x_0}^{x}\left(\frac{\dfrac{\mathrm{d}C_{g_0}}{\mathrm{d}x} + \dfrac{3}{2}\dfrac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}\right)\mathrm{d}x\right],$$

while the appropriate choice for the function Φ_1 is easily shown to be

$$\Phi_1\left(\int_{x_0}^{x} \frac{\mathrm{d}x}{C_{g_0}+U} - t\right) = \tilde{a}_0.$$

The real part of the amplitude of the Stokes wave in the presence of a current is then given by

$$a(x) = \tilde{a}_0 \exp\left[-\int_{x_0}^{x}\left(\frac{\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}\right)\mathrm{d}x\right], \tag{4.8}$$

where the modification (due to the current) to the still-water case is given by the exponential in (4.8). The instability criterion (4.7) involves the unknown function $\beta(x)$. If Taylor-series expansion of $\beta(x)$ about its constant-current value $\tilde{\beta}$ is assumed, and on substitution for A_1, (4.7) can be integrated directly. The eventual expression that is obtained is the requirement that $0 < \cos 2\Sigma\eta \leqslant 1$. When the full spatial variation of $\beta(x)$ is taken into account, the result is not immediately obvious but can be shown to involve the same restriction, namely that $\cos 2\Sigma\eta > 0$.

To further explore the stability of the Stokes wave, (4.5) is multiplied by $a_1\,\mathrm{e}^{-\mathrm{i}\Sigma\eta}$ and the complex-conjugate of (4.5) is subtracted from the resulting expression. The expression obtained is

$$2\mathrm{i}[\beta(x)\,|A_1|^2 - \gamma(x)\,\kappa^2]\,a_1^2 + \mathrm{i}\beta(x)\,|A_1|^2\,[\mathrm{e}^{2\mathrm{i}\Sigma\eta}+\mathrm{e}^{-2\mathrm{i}\Sigma\eta}]\,a_1^2 = 0$$

or equivalently

$$\gamma(x)\,\kappa^2 = \beta(x)\,|A_1|^2\,(1+\cos 2\Sigma\eta)$$
$$\leqslant 2\beta(x)\,|A_1|^2,$$

for $\cos 2\Sigma\eta > 0$. Substitution for γ, β and A_1 then, finally, shows that Stokes waves on a current will be unstable to perturbations with wavenumber $K_0 = \kappa/k_0$ in the range

$$0 < K_0 \leqslant 2\sqrt{2}\,k_0\,\tilde{a}_0 \exp\left[-\int_{x_0}^{x}\left(\frac{\frac{1}{2}\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{4}\frac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}\right)\mathrm{d}x\right]. \tag{4.9}$$

Note that when $U \rightarrow \tilde{U}$ (or $U \rightarrow 0$), since $k_0(x) \rightarrow \tilde{k}_0$, (4.9) reduces to the still-water sideband instability criterion as given by Benjamin–Feir (1967). The derivation is fairly simple and does not appear to have been published before. An alternative approach would be that as given by Yuen & Lake (1978). With their approach, the dispersion relation for the perturbation that is obtained is

$$\Omega^2 = \frac{\sigma_0^2\kappa^2}{8k_0^2}\left\{\frac{\kappa^2}{8k_0^2} - k_0^2\tilde{a}_0^2 \exp\left[-\int_{x_0}^{x}\left(\frac{\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{2}\frac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}\right)\mathrm{d}x\right]\right\}. \tag{4.10}$$

Instability will occur when $\Omega^2 < 0$, and the condition (4.9) is recovered.

A better understanding of the influence of the current on the instability criterion can be obtained by substitution from known wave–current interaction results. For a steady wavetrain moving in the x-direction on a current $U(x)$, the linear theory, deepwater relation for the variation in phase velocity and wavenumber was calculated by Longuet-Higgins & Stewart (1961). With the notation $c_0 = \tilde{c}_0$ when $U = 0$, it is

$$c_0 = \tfrac{1}{2}\tilde{c}_0\left[1+\left(1+\frac{4U}{\tilde{c}_0}\right)^{\frac{1}{2}}\right],$$

$$k_0 = \left(\frac{\tilde{c}_0}{c_0}\right)^2 \tilde{k}_0 = \frac{4\tilde{k}_0}{\left(1+\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{2}}\right)^2}.$$

Since $C_{g_0} = \frac{1}{2}c_0$ in deep water, and for the lower limit of integration $x_0 = 0$, the exponential in (4.9) can be integrated. The instability criterion (4.9) then becomes

$$0 < \frac{\kappa}{k_0} \leqslant \frac{4\sqrt{2}\,k_0\,\tilde{a}_0}{\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{4}}\left(1+\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{2}}\right)}, \tag{4.11}$$

where the arbitrary constant generated in the integration was determined from the condition that the exponential in (4.9) does not contribute to the instability criterion as $U \rightarrow 0$. Substitution for the spatial variation of $k_0(x)$ in (4.11) now gives the desired Benjamin–Feir instability criterion for Stokes waves on a current $U = [U(x), 0, 0]$:

$$0 < \frac{\kappa}{\tilde{k}_0} = \tilde{K}_0 \leqslant \frac{64\sqrt{2}\,\tilde{k}_0\,\tilde{a}_0}{\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{4}}\left(1+\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{2}}\right)^5}. \tag{4.12}$$

With these substitutions we infer from (4.10) that instability of the wavetrain will result if the initial wave steepness $\tilde{\epsilon}_0 = \tilde{k}_0\tilde{a}_0$ is sufficiently large, that is when

$$\tilde{\epsilon}_0 > \frac{\sqrt{2}\tilde{K}_0}{128}\left\{\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{4}}\left(1+\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{2}}\right)^5\right\}. \tag{4.13}$$

This is shown graphically in figure 1 (a, b) where the normalized sideband growth rate $4\pi\,\mathrm{Im}\,(\tilde{\Omega}_0)$, where $\tilde{\Omega}_0 = \Omega/\tilde{\sigma}_0$, is plotted as a function of the steepness $\tilde{\epsilon}_0$ for $\tilde{K}_0 = 0.2$ and 0.4 respectively, and for different values of $\Gamma_1 = U/\tilde{c}_0$. For comparison the still-water Benjamin–Feir growth curve ($\Gamma_1 = 0$) is also shown. The graphs were terminated at $\tilde{\epsilon}_0 = 0.4$ since the waves are expected to break at approximately this value (Michell 1893). It is clear from these figures that for positive current gradients ($\Gamma_1 > 0$), when the waves and the current are propagating in the same direction, waves of greater steepness than in the still-water case are needed before the onset of the instability. Alternatively, for fixed $\tilde{\epsilon}_0$, an increase in Γ_1 leads to a decrease in the sideband growth rates of the waves. Steeper waves are thus needed before instability results, and once instability is reached, the current has a further damping effect on the growth of the sidebands. For an adverse current gradient, when $\Gamma_1 < 0$, the energy density of the waves increases as a result of the radiation stress. The associated rapid steepening of the waves then causes even waves of very gentle initial slope to become unstable, and with a much increased rate of growth of the sidebands. The above results are also clear from the predicted e-folding time of the instability:

$$\tau = \frac{2}{\sigma_0 k_0^2 \tilde{a}_0^2}\exp\left[+\int_{x_0}^{x}\left(\frac{\frac{\mathrm{d}C_{g_0}}{\mathrm{d}x}+\frac{3}{2}\frac{\mathrm{d}U}{\mathrm{d}x}}{C_{g_0}+U}\right)\mathrm{d}x\right]$$

$$= \frac{2}{\tilde{\sigma}_0 \tilde{k}_0^2 \tilde{a}_0^2}\,\frac{\left[1+\frac{4U}{\tilde{c}_0}\right]^{\frac{1}{4}}\left[1+\left(1+\frac{4U}{\tilde{c}_0}\right)^{\frac{1}{2}}\right]^7}{128}.$$

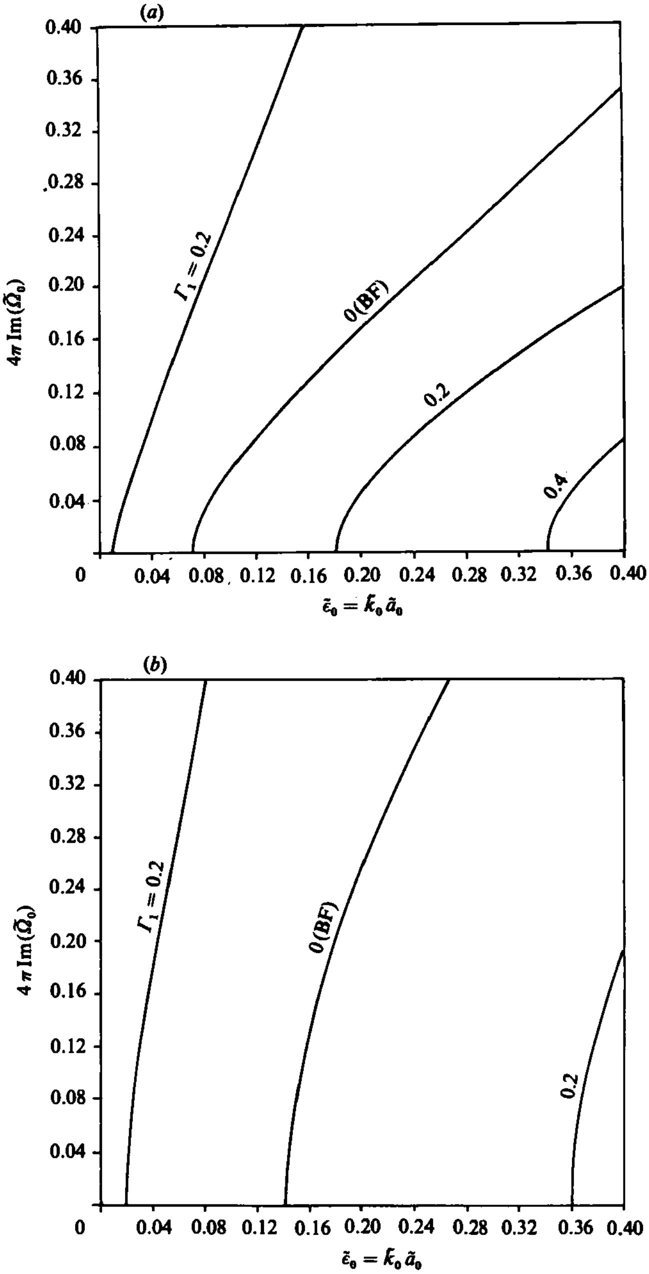

FIGURE 1. The normalized sideband growth rate $4\pi \operatorname{Im}(\tilde{\Omega}_0)$ as a function of the initial wave steepness $\tilde{\epsilon}_0 = \tilde{k}_0 \tilde{a}_0$, for various values of $\Gamma_1 = U/\tilde{c}_0$, and for the modulation wavenumber (*a*) $\tilde{K}_0 = 0.2$, (*b*) 0.4.

For $\Gamma_1 > 0$ a longer time than in the still-water case is required for the same amount of growth of the sidebands, while for $\Gamma_1 < 0$ a shorter e-folding time is predicted.

The results of this section are only expected to apply in the special case of waves and current propagating in the same (or opposite) directions. Physically this corresponds to the situation of waves generated on a current, with no shear, and with the waves propagating along with (or against) the current. For waves swept along by (or propagating against) a large geostrophic current system, such as the Gulf Stream or Agulhas Current, since shear is not considered, the results will only apply to waves locally generated on the current.

4.2. *Steady current, varying across the stream*

The form of the MCSE in the presence of a shearing current $U = (0, V(x, 0)$ was given by (3.16). When the results of §4.1 are used, the Benjamin–Feir instability criterion for (3.16) can be inferred directly provided irrotational wave perturbations can be justified in the presence of a vortical mean current. With the scaling introduced in §3 the current vorticity is of order $\epsilon \Delta \pi$ and thus the rotational part of the wave motion, due to the distortions of the vortex lines, is order $\epsilon \Delta \pi$ smaller than the irrotational part. This is sufficient to justify the following analysis. Of course, the assumption of irrotationality for the mean current, as motivated in §2, cannot be justified.

The instability criterion (4.9) could also have been written as

$$0 < K_0 \leqslant 2\sqrt{2}\, k_0 a_0, \tag{4.14}$$

which is just the original Benjamin–Feir (1967) result, but with the difference that K_0, k_0 and a_0 in (4.14) are now functions of $U(x)$, which must be determined. The spatial dependence of $a_0(x)$ was given by (4.8), which is the real solution to the linear and steady form of (4.3). With reference to (4.1), the real solution to the steady, linear asymptotic Stokes wave equation (i.e. when $\Delta \ll \epsilon, \Delta \ll \pi$) is needed. For (3.16) this is equivalent to seeking the solution to

$$C_{g_0} \cos\phi \frac{\mathrm{d}a_0}{\mathrm{d}x} + \left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(C_{g_0}\cos\phi) + \tfrac{1}{4}\sin\phi\cos\phi\frac{\mathrm{d}V}{\mathrm{d}x}\right]a_0 = 0,$$

and the spatial dependence of $a_0(x)$ is easily found as

$$a_0(x) = \tilde{a}_0 \exp\left\{-\int_{x_0}^{x}\left[\frac{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(C_{g_0}\cos\phi) + \frac{1}{4}\sin\phi\cos\phi\frac{\mathrm{d}V}{\mathrm{d}x}}{C_{g_0}\cos\phi}\right]\mathrm{d}x\right\}. \tag{4.15}$$

The lowest-order spatial variations in phase velocity, wavenumber and angle ϕ between the wave propagation direction and x-axis are also known from the work of Longuet-Higgins & Stewart (1961). They are given by

$$c_0 = \frac{\tilde{c}_0}{1-\left(\dfrac{V}{\tilde{c}_0}\right)\sin\tilde{\phi}},$$

$$k_0 = \tilde{k}_0\left[1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^2,$$

$$\sin\phi = \frac{\sin\tilde{\phi}}{\left[1-\left(\dfrac{V}{\tilde{c}_0}\right)\sin\tilde{\phi}\right]^2}.$$

Substitution of these relations, and performing the integration (where the lower limit $x_0 = 0$), gives for the criterion (4.14)

$$0 < \tilde{K}_0 \leqslant \frac{2\sqrt{2}\,\tilde{k}_0\tilde{a}_0\cos^{\frac{1}{2}}\tilde{\phi}\left[1-\dfrac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^6}{\left[\left(1-\dfrac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^4 - \sin^2\tilde{\phi}\right]^{\frac{1}{4}}}. \tag{4.16}$$

The constant of integration was determined from the condition that the exponential in (4.15) does not contribute to the instability criterion as $V \to 0$.

The initial wave steepness $\tilde{\epsilon}_0 = \tilde{k}_0 \tilde{a}_0$ required for instability can also be calculated for a shear current $\boldsymbol{U} = [0, V(x), 0]$. It is

$$\tilde{\epsilon}_0 > \frac{\sqrt{2}\,\tilde{K}_0\left[\left(1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^4 - \sin^2\tilde{\phi}\right]^{\frac{1}{4}}}{4\cos^{\frac{1}{2}}\tilde{\phi}\left[1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^6}. \tag{4.17}$$

From (4.17) it is clear that there is an upper limit to V for which a solution exists:

$$\frac{V}{\tilde{c}_0} \leqslant \frac{1-\sin^{\frac{1}{2}}\tilde{\phi}}{\sin\tilde{\phi}}, \tag{4.18}$$

a relation also given by Longuet-Higgins & Stewart (1961). At the upper limit, for $V > 0$, and as $\phi \to 90°$, the waves are reflected by the current. For $V < 0$, however, as the opposing velocity of the current increase, the angle ϕ will decrease and the direction of propagation of the waves will tend to become normal to the current. In the limit $\tilde{\phi} = 0°$, (4.17) gives

$$\tilde{\epsilon}_0 > \frac{\sqrt{2}\,\tilde{K}_0}{4}, \tag{4.19}$$

the still-water Benjamin–Feir criterion, and as expected, the current has no effect on the instability criterion.

When (4.17) is compared with the still-water Benjamin–Feir criterion, (4.19), the influence of the shear current on the instability criterion can be identified as

$$F\left(\frac{V}{\tilde{c}_0}\right) = \frac{\left[\left(1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^4 - \sin^2\phi\right]^{\frac{1}{4}}}{\cos^{\frac{1}{2}}\tilde{\phi}\left[1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^6}. \tag{4.20}$$

The function $F(V/\tilde{c}_0)$ is shown graphically in figure 2 for various angles of initial entry $\tilde{\phi}$. Values of $F(V/\tilde{c}_0) > 1.0$ then indicate a larger initial steepness required for instability and thus greater stability than would be found in the still-water case, while for $F(V/\tilde{c}_0) < 0$ the waves will be relatively more unstable in the Benjamin–Feir sense. It is interesting to note that only waves with an initial angle of entry of less than 45° can exhibit greater stability when $V/\tilde{c}_0 > 0$. Thus for small $\tilde{\phi}$, and when the current and wave propagation directions coincide, greater stability of the waves is predicted. For waves opposing the current, however, an approximate angle of $\tilde{\phi} > 65°$ is necessary before greater stability than in still water will result. These results are also reflected in the variation of the e-folding time of the instability:

$$\begin{aligned}\tau &= \frac{2}{\sigma_0 k_0^2 a_0^2} \\ &= \frac{2}{\tilde{\sigma}_0 \tilde{k}_0^2 \tilde{a}_0^2}\,\frac{\left\{\left[1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^4 - \sin^2\tilde{\phi}\right\}^{\frac{1}{2}}}{\cos\tilde{\phi}\left[1-\frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^9}.\end{aligned}$$

When the current-induced factor on the right of this expression is plotted as a function of $V/\tilde{c}_0$, a graph very similar to figure 3 is obtained. For $\tilde{\phi} \leqslant 45°$ and $V/\tilde{c}_0 > 0$, a longer e-folding time than in the still-water case is predicted. The waves

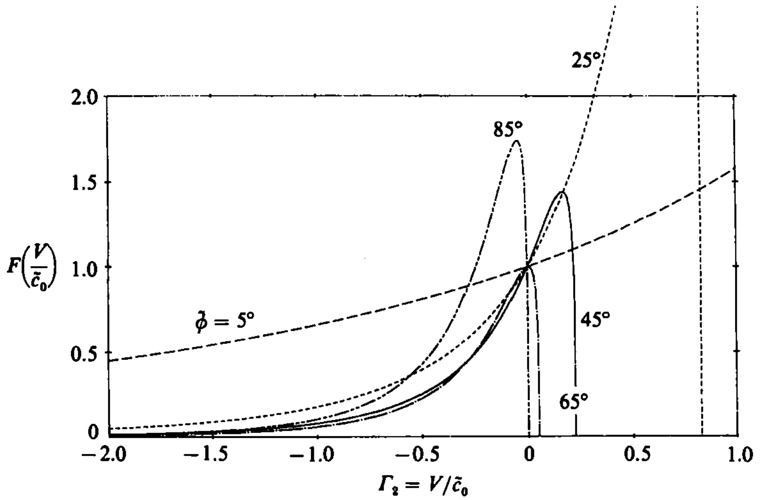

FIGURE 2. The factor $F(V/\tilde{c}_0)$ for various angles of initial entry $\tilde{\phi}$.

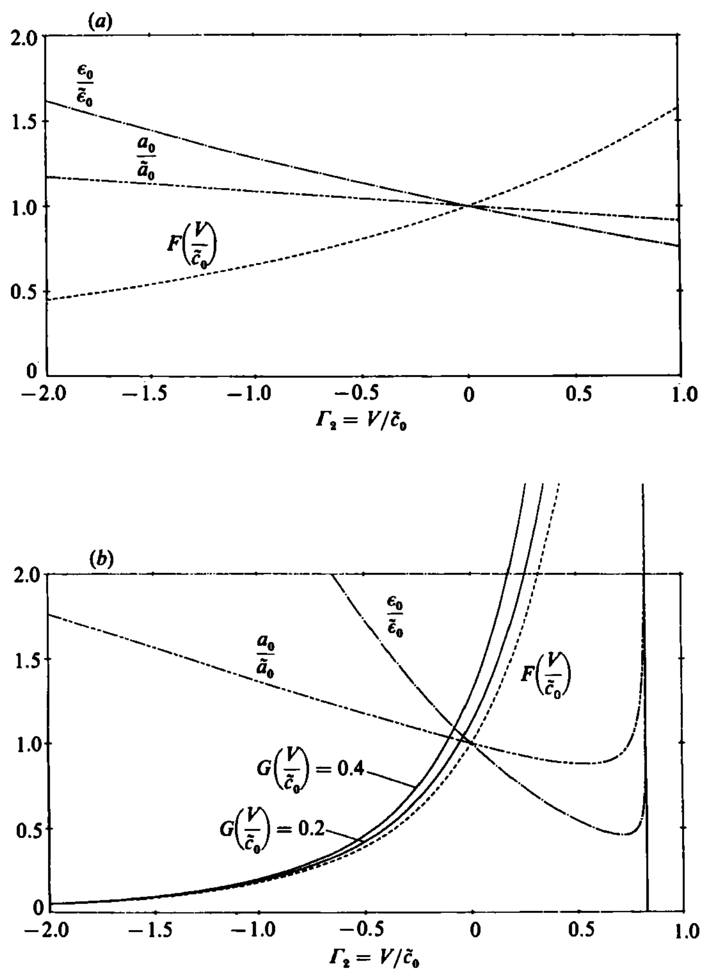

FIGURE 3(*a*, *b*). For caption see facing page.

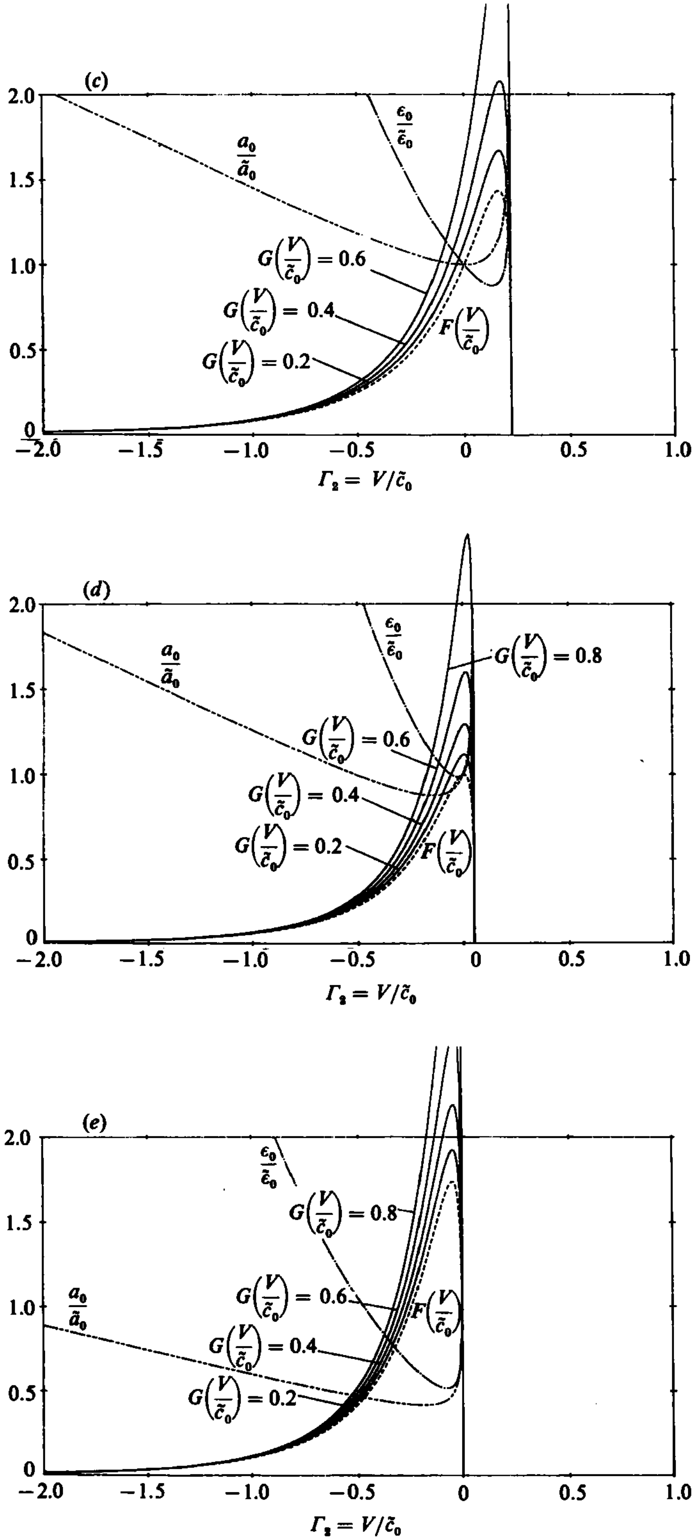

FIGURE 3. The factor $F(V/\tilde{c}_0)$, the relative amplitude $a_0/\tilde{a}_0$ and the relative steepness $\epsilon_0/\tilde{\epsilon}_0$ as a function of $V/\tilde{c}_0$, and for angles of incidence (*a*) $\tilde{\phi} = 5°$; (*b*) 25°; (*c*) 45°; (*d*) 65°; (*e*) 85°.

are more stable. Similarly, when $V/\tilde{c}_0 < 0$, an angle $\tilde{\phi}$ of approximately 65° is necessary before a longer e-folding time, and thus greater stability, is predicted. These results are not unexpected as can be seen from figure 3 which shows the relative amplitude (Longuet-Higgins & Stewart 1961)

$$\frac{a_0}{\tilde{a}_0} = \left[\frac{\sin 2\tilde{\phi}}{\sin 2\phi}\right]^{\frac{1}{2}}$$

and the relative steepness

$$\frac{\epsilon_0}{\tilde{\epsilon}_0} = \frac{k_0 a_0}{\tilde{k}_0 \tilde{a}_0} = \left[1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^2 \left[\frac{\sin 2\tilde{\phi}}{\sin 2\phi}\right]^{\frac{1}{2}}$$

of the waves, together with the factor $F(V/\tilde{c}_0)$, as a function of $V/\tilde{c}_0$ and for various angles of initial entry $\tilde{\phi}$ (dashed lines only). It is clear that greater stability ($F(V/\tilde{c}_0) > 1$) can only result when $\epsilon_0/\tilde{\epsilon}_0 < 1$. As the steepness of the waves (relative to the still-water situation) is reduced owing to the current influence, greater stability of the waves results. The opposite also follows – as the waves steepen on an opposing current, the initial steepness required for instability is reduced, and the waves will be relatively more unstable.

The upper limit of $\Gamma_2 = V/\tilde{c}_0 > 0$ is given by (4.18) and is reflected in figures 2 and 3 as the vertical line at $\Gamma_{2(\text{crit})}$, where $\Gamma_{2(\text{crit})}$ is the value of Γ_2 for which no solution can exist as $\Gamma_2 > \Gamma_{2(\text{crit})}$. At this value $\Gamma_2 = \Gamma_{2(\text{crit})}$ the wave propagation direction becomes parallel to the current direction, resulting in reflection of the waves and a rapid increase in the wave amplitude and steepness in the vicinity of the caustic. It is clear from these figures that the position of greatest instability of the waves must also be in the vicinity of the caustic. This is contrary to the results of Smith (1976) who found that steady finite-amplitude waves are stable at a reflection caustic.

At this point the results of this section can easily be applied to the still-water, fourth-order Benjamin–Feir instability criterion as calculated by Dysthe (1979). Janssen (1983) showed that the wave-induced mean flow described by the Dysthe equation could significantly explain the difference between the approximate results of Benjamin & Feir (and as reproduced by the cubic Schrödinger equation) and the exact computer-generated results of Longuet-Higgins (1978), provided $\tilde{\epsilon}_0 < 0.25$. More accurate results are then also expected for this analysis. The analysis of Dysthe gives the wave steepness $\tilde{\epsilon}_0$ required for instability as

$$\tilde{\epsilon}_0 > \frac{\surd 2\,\tilde{K}_0}{4(1-|\tilde{K}_0|)^{\frac{1}{2}}},$$

where the effect of the wave-induced current is given by the modulus-of-$\tilde{K}_0$ term. In the presence of a shear current the fourth-order contribution becomes

$$\left(1 - \frac{\tilde{K}_0}{\left[1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^2}\right)^{-\frac{1}{2}}, \tag{4.21}$$

and (4.17) can be written as

$$\tilde{\epsilon}_0 > \frac{\surd 2\,\tilde{K}_0\left[\left(1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^4 - \sin^2\tilde{\phi}\right]^{\frac{1}{4}}}{4\cos^{\frac{1}{2}}\tilde{\phi}\left[1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^5\left[\left(1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^2 - \tilde{K}_0\right]^{\frac{1}{2}}}. \tag{4.22}$$

Since only real vales of $\tilde{\epsilon}_0$ are possible, this introduces a range of values for which $\tilde{K}_0 = \kappa/\tilde{k}_0$ is meaningful. The restriction

$$\tilde{K}_0 < \left(1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^2 \tag{4.23}$$

is in fact only significant for $V/\tilde{c}_0 \sin\tilde{\phi}$ greater than zero, and as a consequence requires $\tilde{K}_0$ to be small. An upper limit for $V/\tilde{c}_0$ is already known from (4.18), so that (4.23) becomes

$$0 < \tilde{K}_0 \sin\tilde{\phi},$$

and it is clear that with decreasing $\tilde{\phi}$, $\tilde{K}_0$ will also be restricted to a smaller range.

The above results, as predicted by the fourth-order Dysthe equation, can also be compared to the still-water Benjamin–Feir instability criterion, and in the following manner. Comparison of (4.22) with (4.19) identifies the influence of the shear current as

$$G\left(\frac{V}{\tilde{c}_0}\right) = \frac{\left[\left(1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^4 - \sin^2\tilde{\phi}\right]^{\frac{1}{4}}}{\cos^{\frac{1}{2}}\tilde{\phi}\left[1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right]^5 \left[\left(1 - \frac{V}{\tilde{c}_0}\sin\tilde{\phi}\right)^2 - \tilde{K}_0\right]^{\frac{1}{2}}}.$$

The function $G(V/\tilde{c}_0)$ is also shown in figure 3 (solid lines) for various values of $\tilde{K}_0$. Since $\sin 5° < 0.2$, no values of $G(V/\tilde{c}_0)$ are shown in figure 3(*a*). The influence of the wave-induced current, for fixed values of Γ_2, can be seen to cause greater stability of the waves in the Benjamin–Feir sense. That is, waves of greater steepness than that predicted by the cubic Schrödinger equation are required before the instability will result.

In figure 4(*a*, *b*) the normalized sideband growth rate $4\pi\,\mathrm{Im}\,(\tilde{\Omega}_0)$ is shown as a function of the initial wave steepness $\tilde{\epsilon}_0$ for angles of initial incidence of 45° and 85°, and for $\tilde{K}_0 = 0.2$. It can be seen that, whereas the growth rates are relatively insensitive to the value of Γ_2 when $\tilde{\phi} = 45°$, the opposite is true when $\tilde{\phi} = 85°$. The initial steepness values required for instability increase (decrease) dramatically for a very small increase (decrease) in Γ_2, and the growth curves reflect this by a large increase (decrease) in the predicted growth rates, when the steepness $\tilde{\epsilon}_0$ is kept fixed.

4.3. *Long-wave and short-wave interactions*

The results of §§4.1 and 4.2 are also expected to apply for the situation of internal waves, or long (deep-water) surface waves, interacting with short surface waves. Here, however, the magnitude of the velocity field is expected to be considerably smaller. To first order, since the particle orbits of the long wave are circular, the effect of the long wave on the short wave will probably be zero. To second order, however, the fluid particle velocities under the long wave have a non-zero mean during a wave cycle. Phillips (1977) gives this Lagrangian drift velocity as

$$u_{\mathrm{L}} = \frac{\sigma_1 k_1 a_1^2}{2}\frac{\cosh 2k_1(z+d)}{\sinh^2 k_1 d},$$

where the subscript 1 is used to indicate the long wave. In deep water, when $k_1 d \gg 1$, this is

$$u_{\mathrm{L}} = \sigma_1 k_1 a_1^2 = \epsilon_1^2 c_1.$$

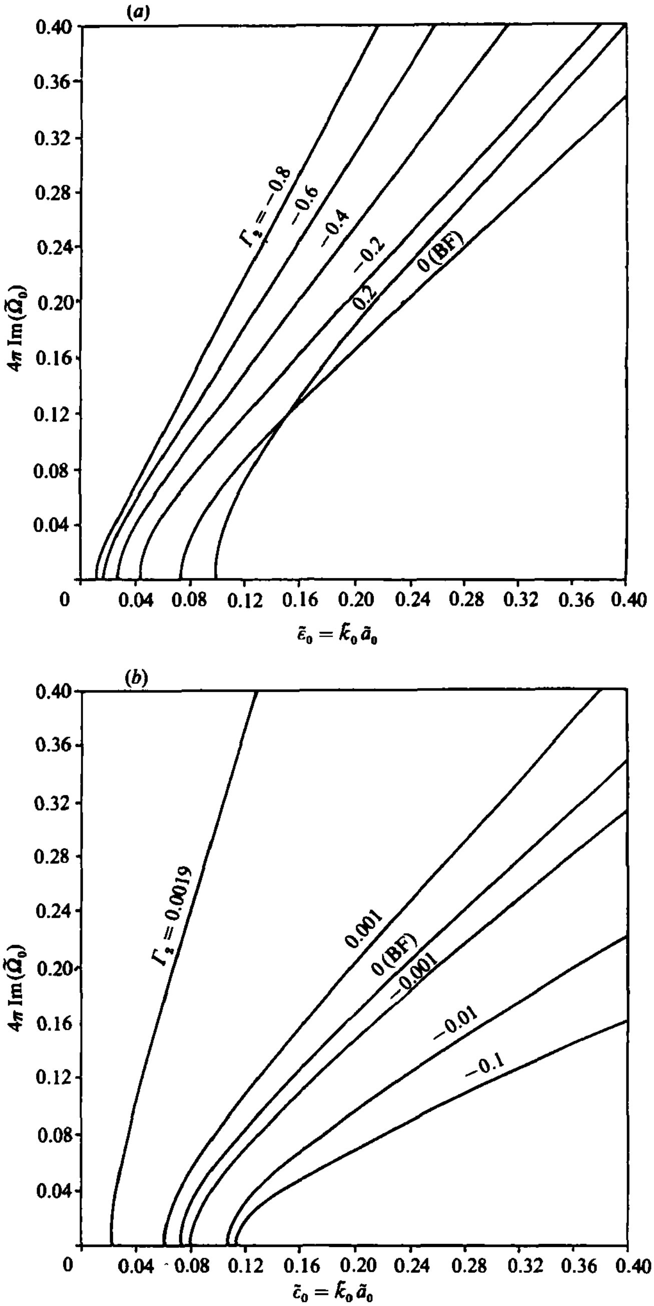

FIGURE 4. The normalized sideband growth rate $4\pi\,\mathrm{Im}(\tilde{\Omega}_0)$ as a function of the initial wave steepness $\tilde{\epsilon}_0 = \tilde{k}_0 \tilde{a}_0$, for various values of $\Gamma_2 = V/\tilde{c}_0$, and for angles of incidence (*a*) $\tilde{\phi} = 45°$; (*b*) 85°. The modulation wavenumber $\tilde{K}_0 = 0.2$.

This means that the results (4.13), (4.17) and (4.22) can be applied directly provided $\Gamma_1 = U/\tilde{c}_0$ and $\Gamma_2 = V/\tilde{c}_0$ in these relations are replaced by

$$\Xi = \epsilon_1^2 \frac{c_1}{\tilde{c}_0} = \epsilon_1^2 \frac{T_1}{\tilde{T}_0},$$

and where the subscript 0 is understood to indicate the short wave.

5. Concluding remarks

In order to investigate the influence of a one-dimensional current on the third-order Benjamin–Feir instability of deep-water, finite-amplitude surface waves, a cubic Schrödinger equation in the presence of a current was derived. The approach of Yuen & Lake (1975) was chosen in preference to the more complicated multiple-scales technique. This necessitated the postulation and verification of an appropriate Lagrangian for waves on a current. The Lagrangian (2.1) was shown to generate the correct governing equation, as well as the relevant boundary conditions. Manipulation of the averaged Lagrangian (2.8) generated the desired amplitude equation, while the dispersion relation to the desired order was obtained from the variation of (2.9).

The derivation of the MCSE was found to involve three ordering parameters: the wave slope ϵ; the spectral bandwidth Δ and a third parameter denoting the ratio of the group length to the current length. The MCSE, when applied to a steady current with variations along the stream, gives the modified Benjamin–Feir instability criterion as (4.9). On substitution of known first-order results in this expression, the initial wave steepness required for instability was found to be a function of the ratio of the current velocity U and the still-water phase velocity of the waves: $\Gamma_1 \equiv U/\tilde{c}_0$. For waves propagating in the same direction as the current, the current was found to have a stabilizing effect on the waves. The waves are stretched by the current so that smaller steepness values result, while, simultaneously, a longer time than in the still-water case is required for the same amount of growth of the sidebands. For an adverse current gradient, a rapid destabilization of the waves was predicted. This is due to the steepening effect of the current, as well as the shorter time required for an equivalent amount of growth of the sidebands, when compared with the still-water case.

For waves on a shearing current, the instability criterion is (4.14), with the spatial dependence of $a_0(x)$ given by (4.15). In this case the initial wave steepness required for instability was found to be a function of the angle of incidence $\tilde{\phi}$, as well as a function of the parameter $\Gamma_2 = V/\tilde{c}_0$. For $\Gamma_2 > 0$ and an angle of incidence $\tilde{\phi} \leqslant 45°$, and excluding values of Γ_2 in the vicinity of the caustic, the waves will be more stable in the Benjamin–Feir sense. For $\tilde{\phi}$ marginally smaller than 90°, i.e. in the neighbourhood of the caustic, rapid destabilization of the waves is predicted owing to the amplification in wave steepness. For $\Gamma_2 > 0$ and $\tilde{\phi} > 45°$, values of Γ_2 are close to the caustic regime, and small steepness values (and with corresponding short e-folding timescales) are required for instability. The waves are thus very unstable when compared to the still-water case. For $\Gamma_2 < 0$ and $\tilde{\phi}$ less than approximately 65°, greater instability of the waves is predicted. For $\tilde{\phi} > 65°$, as the steepness of the waves decrease, greater stability than for waves on still water is expected. When the above results are repeated using the fourth-order results of Dysthe (1979), it is found that, for a fixed value of Γ_2, the contribution of the wave-induced current is to further stabilize the waves (when compared to the third-order results). The upper limit $\tilde{K}_0 < \sin\tilde{\phi}$, was also identified.

When the results of this paper are generalized to long-wave and short-wave interactions, the current velocity must be replaced by the second-order Stokes drift velocity associated with the long wave. Owing to the small magnitude of the Stokes drift current, rather large steepness values of the long wave are required for real-life applications.

The author is pleased to acknowledge useful discussions with Professor O. M. Phillips.

REFERENCES

BENJAMIN, T. B. & FEIR, J. E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27**, 417–430.

DJORDJEVIC, V. D. & REDEKOPP, L. G. 1978 On the development of a packet of surface gravity waves moving over an uneven bottom. *Z. angew. Math. Phys.* **29**, 950–962.

DYSTHE, K. B. 1979 Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proc. R. Soc. Lond.* A **369**, 105–114.

JANSSEN, P. A. E. M. 1983 On a fourth-order envelope equation for deep-water waves. *J. Fluid Mech.* **126**, 1–11.

LAKE, B. M., YUEN, H. C., RUNGALDIER, H. & FERGUSON, W. E. 1977 Nonlinear deep-water waves: theory and experiment. Part 2. Evolution of a continuous wave train. *J. Fluid Mech.* **83**, 49–74.

LIGHTHILL, M. J. 1965 Contributions to the theory of waves in non-linear dispersive systems. *J. Inst. Maths. Applics.* **1**, 269–306.

LONGUET-HIGGINS, M. S. 1978 The instabilities of gravity waves of finite amplitude in deep water. II. Subharmonics. *Proc. R. Soc. Lond.* A **360**, 489–505.

LONGUET-HIGGINS, M. S. & STEWART, R. W. 1961 The changes in amplitude of short gravity waves on steady non-uniform currents. *J. Fluid Mech.* **10**, 529–549.

LONGUET-HIGGINS, M. S. & STEWART, R. W. 1964 Radiation stresses in water waves: a physical discussion, with applications. *Deep-Sea Res.* **11**, 529–562.

LUKE, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

MICHELL, J. H. 1893 The highest waves in water. *Phil. Mag.* **36**, 430–437.

PEREGRINE, D. H. 1976 Interaction of water waves and currents. *Adv. Appl. Mech.* **16**, 10–117.

PHILLIPS, O. M. 1977 *The Dynamics of the Upper Ocean*. Cambridge University Press.

PHILLIPS, O. M. 1981 Wave interactions – the evolution of an idea. *J. Fluid Mech.* **106**, 215–227.

SMITH, R. 1976 Giant waves. *J. Fluid Mech.* **77**, 417–431.

STUART, J. T. & DIPRIMA, R. C. 1978 The Eckhaus and Benjamin–Feir resonance mechanisms. *Proc. R. Soc. Lond.* A **362**, 27–41.

TURPIN, F-M., BENMOUSSA, C. & MEI, C. C. 1983 Effects of slowly varying depth and current on the evolution of a Stokes wavepacket. *J. Fluid Mech.* **132**, 1–23.

WEST, B. J. 1981 *Deep water gravity waves (weak interaction theory)*. Lecture Notes in Physics, vol. 146. Springer.

WHITHAM, G. B. 1960 A note on group velocity. *J. Fluid Mech.* **9**, 347–352.

WHITHAM, G. B. 1962 Mass, momentum and energy flux in water waves. *J. Fluid Mech.* **12**, 135–147.

WHITHAM, G. B. 1965 A general approach to linear and non-linear dispersive waves using a Lagrangian. *J. Fluid Mech.* **22**, 273–283.

WHITHAM, G. B. 1974 *Linear and Non-Linear Waves*. Wiley-Interscience.

YUEN, H. C. & LAKE, B. M. 1975 Non-linear deep water waves: theory and experiment. *Phys. Fluids* **18**, 956–960.

YUEN, H. C. & LAKE, B. M. 1978 Non-linear wave concepts applied to deep water waves. In *Solitons in Action*, pp. 89–126. Academic.